MS221 EBA

The Open University

EXPLORING MATHEMATICS

Exercise Booklet A

Exercise Booklet

A

1.4

SUP 49260 2

Contents

The exercises in this booklet are intended to give further practice, should you require it, in handling the main mathematical ideas in each chapter of MS221, Block A. The exercises are ordered by chapter and section, and are numbered correspondingly: for example, Exercise 5.2 for Chapter A1 is the second exercise on Section 5 of that chapter.

Exercises for Chapter A1

Section 1

Exercise 1.1

Rationalise the denominator of each of the following fractions.

(a) $\dfrac{1}{1-\sqrt{5}}$

(b) $\dfrac{\sqrt{2}}{\sqrt{17}+\sqrt{2}}$

Exercise 1.2

For each of the following quadratic equations, find the sum of the solutions, the product of the solutions, and the sum of the cubes of the solutions, without solving the equation. (Use the identity $(\alpha+\beta)^3 = \alpha^3 + 3\alpha\beta(\alpha+\beta) + \beta^3$.)

(a) $x^2 + 5x - 7 = 0$

(b) $5x^2 - 9x + 1 = 0$

Section 3

Exercise 3.1

Find a closed form for each of the following sequences.

(a) $u_0 = 3,\ u_1 = 7,$
$u_{n+2} = 5u_{n+1} + 6u_n \quad (n = 0, 1, 2, \ldots)$

(b) $u_0 = 5,\ u_1 = 20,$
$u_{n+2} = -8u_{n+1} - 16u_n \quad (n = 0, 1, 2, \ldots)$

(c) $u_0 = 4,\ u_1 = 6,$
$u_{n+2} = 1.8u_{n+1} - 0.77u_n \quad (n = 0, 1, 2, \ldots)$

Section 5

Exercise 5.1

(a) Show, by substitution, that the sequence

$$u_n = 7^n - (-5)^n \quad (n = 0, 1, 2, \ldots)$$

satisfies the recurrence system

$$u_0 = 0,\ u_1 = 12,$$
$$u_{n+2} = 2u_{n+1} + 35u_n \quad (n = 0, 1, 2, \ldots).$$

(b) Show, by substitution, that the sequence u_n in part (a) satisfies the Cassini-type identity

$$u_{n-1}u_{n+1} - u_n^2 = -144(-35)^{n-1},$$

for $n = 1, 2, 3, \ldots$.

Exercise 5.2

(a) Show, by substitution, that the solution to Exercise 3.1(a) satisfies the identity:

$$u_{n-1}u_{n+1} - u_n^2 = 110(-6)^{n-1},$$

for $n = 1, 2, 3, \ldots$.

(b) Show, by substitution, that the solution to Exercise 3.1 (b) satisfies the identity:

$$u_{n-1}u_{n+1} - u_n^2 = -100 \times 16^n,$$

for $n = 1, 2, 3, \ldots$.

Exercise 5.3

(a) Use the Fibonacci recurrence relation to prove that

$$\frac{1}{F_{n+1}} - \frac{1}{F_{n+2}} = \frac{F_n}{F_{n+1}F_{n+2}},$$

for $n = 0, 1, 2, \ldots$.

(b) Use part (a) and telescoping cancellation to deduce the identity

$$\frac{F_0}{F_1F_2} + \frac{F_1}{F_2F_3} + \cdots + \frac{F_n}{F_{n+1}F_{n+2}} = 1 - \frac{1}{F_{n+2}},$$

for $n = 0, 1, 2, \ldots$.

Exercises for Chapter A2

Section 2

Exercise 2.1

Rearrange each of the following equations in the form of the equation $x^2/a^2 + y^2/b^2 = 1$ and sketch the corresponding ellipse.

(a) $9x^2 + 49y^2 = 441$

(b) $6x^2 + 12y^2 - 3 = 0$

Exercise 2.2

Rearrange each of the following equations in the form of the equation $x^2/a^2 - y^2/b^2 = 1$ and sketch the corresponding hyperbola.

(a) $9x^2 - 25y^2 = 225$

(b) $3y^2 + 12 - 36x^2 = 0$

Exercise 2.3

Rearrange the following equation in the form of the equation $y^2 = 4ax$ and sketch the corresponding parabola.

$$16x - 8y^2 = 0.$$

Exercise 2.4

Write down the equations of the following conics, given that each is in standard position.

(a) The ellipse with vertices $(11, 0)$, $(-11, 0)$, $(0, 10)$, $(0, -10)$.

(b) The hyperbola with vertices $(5, 0)$, $(-5, 0)$ and asymptotes $y = \pm 2x$.

(c) The parabola including the point $(1, -6)$.

Section 3

Exercise 3.1

(a) For each of the following ellipses, find the foci, directrices and eccentricity, and mark the foci and directrices on a sketch of the ellipse. (You sketched these ellipses in Exercise 2.1 above.)

(i) $9x^2 + 49y^2 = 441$

(ii) $6x^2 + 12y^2 - 3 = 0$

(b) The points $(\frac{1}{\sqrt{2}}, 0)$ and $(0, -\frac{1}{2})$ are on the ellipse $6x^2 + 12y^2 - 3 = 0$. You have found the foci F, F', directrices d, d' and eccentricity e of this ellipse in part (a)(ii) above. Verify that the properties $PF = ePd$ and $PF' = ePd'$ both hold at each of these points.

Exercise 3.2

For each of the following hyperbolas, find the foci, directrices and eccentricity. (You sketched these hyperbolas in Exercise 2.2 above.)

(a) $9x^2 - 25y^2 = 225$

(b) $3y^2 + 12 - 36x^2 = 0$

Exercise 3.3

Find the focus and the directrix of the following parabola and indicate them on a sketch of the parabola. (You sketched this parabola in Exercise 2.3 above.)

$$16x - 8y^2 = 0$$

Exercise 3.4

For each of the following conics in standard position, you are given some of the information about its foci, directrices and eccentricity. Use the fact that the conic is in standard position to find the equation of the conic.

(a) The conic with a focus at $(6, 0)$ and eccentricity 0.6.

(b) The conic with a directrix $x = \frac{1}{3}$ and eccentricity 6.

(c) The conic with a focus at $(5, 0)$ and a directrix $x = -5$.

(d) The conic with a focus at $(2, 0)$ and a directrix $x = 8$.

Exercise 3.5

Let P be the point (x, y), F the point $(18, 0)$ and d the line $x = 2$.

(a) Write down expressions for the distances PF and Pd.

(b) Suppose that the point $P(x, y)$ satisfies $PF = 3Pd$. Use your answer to part (a) to find an equation satisfied by $P(x, y)$ and show that this equation can be rearranged to give the equation of a hyperbola in standard position.

Section 4

Exercise 4.1

Classify each of the following quadratic curves as an ellipse, parabola or hyperbola, and then sketch each curve.

(a) $36x^2 - 25y^2 - 72x - 100y - 964 = 0$

(b) $4x^2 + 8y^2 + 4x + 48y + 41 = 0$

(c) $y^2 - 12x + 8y + 40 = 0$

Section 5

Exercise 5.1

Classify each of the following quadratic curves as an ellipse, parabola or hyperbola and then write down a parametrisation for each curve.

(a) $x - 2y^2 = 0$

(b) $3x^2 - 27 + 36y^2 = 0$

(c) $7x^2 - 4y^2 = 1$

Exercise 5.2

Use the solution to Exercise 4.1 to write down parametric equations for each of the following quadratic curves.

(a) $36x^2 - 25y^2 - 72x - 100y - 964 = 0$

(b) $4x^2 + 8y^2 + 4x + 48y + 41 = 0$

(c) $y^2 - 12x + 8y + 40 = 0$

Exercise 5.3

For each of the following conics given by parametric equations, write down the eccentricity of the conic.

(a) $x = 4\cos t, \quad y = \sqrt{7}\sin t \quad (0 \le t \le 2\pi)$

(b) $x = 5t^2, \quad y = 10t$

(c) $x = -7 + 4\sec t, \quad y = 6 + 3\tan t$
$(-\frac{1}{2}\pi < t < \frac{1}{2}\pi, \frac{1}{2}\pi < t < \frac{3}{2}\pi)$.

Exercises for Chapter A3

Section 1

Exercise 1.1

Let K be the hyperbola in standard position

$$\frac{x^2}{3} - \frac{y^2}{2} = 1$$

and L the quadratic curve

$$\frac{(x-5)^2}{3} - \frac{(y+1)^2}{2} = 1.$$

(a) Find the translation function t which maps K onto L.

(b) Find the translation function which maps L onto K.

Exercise 1.2

Write down a parametrisation function for each of the following curves.

(a) the circle $(x+4)^2 + (y-1)^2 = 9$

(b) the ellipse $\dfrac{x^2}{7} + \dfrac{y^2}{3} = 1$

(c) the left branch of the hyperbola $\dfrac{x^2}{4} - \dfrac{y^2}{5} = 1$

Exercise 1.3

This exercise concerns the quadratic curve L with equation

$$x^2 + 2y^2 + 2x - 8y = 7.$$

(a) Find the translation function that maps the quadratic curve L onto a conic K in standard position. Write down the equation of K.

(b) Write down the translation function that maps K onto L.

(c) Find a parametrisation function (or functions) for the quadratic curve L.

Section 2

Exercise 2.1

Express each of the following translations in the form $t_{p,q}$.

(a) The inverse of $t_{-7,9}$.

(b) The composite $t_{0,4} \circ t_{-3,1}$.

Exercise 2.2

Express each of the following rotations in the form r_θ, where θ lies in the interval $(-\pi, \pi]$.

(a) The inverse of $r_{-2\pi/7}$.

(b) The composite $r_{5\pi/6} \circ r_{2\pi/3}$.

Exercise 2.3

Give a definition in two-line notation of each of the following isometries.

(a) A translation nine units to the right and four units down.

(b) A translation which maps the point $(2,4)$ to $(4,2)$.

(c) A clockwise rotation through an angle of $\frac{3}{4}\pi$ about the origin.

(d) A reflection in the line ℓ through the origin making an angle $\frac{3}{8}\pi$ with the positive x-axis.

Exercise 2.4

Determine the rule of each of the following composite isometries.

(a) $t_{1,-2} \circ r_{-3\pi/2}$

(b) $r_{-3\pi/2} \circ t_{1,-2}$

(c) $r_{-\pi/2} \circ q_{\pi/4}$

(d) $q_{\pi/4} \circ r_{-\pi/2}$

Exercise 2.5

Let T be the triangle with vertices $(-2,-1)$, $(-1,1)$ and $(2,0)$.

(a) Calculate the images of these vertices under each of the isometries $t_{-1,3}$, r_π and q_0.

(b) Hence sketch $t_{-1,3}(T)$, $r_\pi(T)$ and $q_0(T)$.

(c) Determine the rule of the composite isometry

$$f = r_\pi \circ t_{-1,3}.$$

(d) Use your answer to part (c) to calculate the images of the vertices of T under f and hence sketch $f(T)$.

Exercise 2.6

Let h be the transformation that reflects the plane in the x-axis and let k be the transformation that reflects the plane in the y-axis.

(a) Give definitions in two-line notation of h and k.

(b) Use your answer to part (a) to determine the composite transformation $h \circ k$ and give a geometric interpretation of $h \circ k$.

Exercise 2.7

Determine the composite $g \circ f$, where

$$\begin{aligned} f\colon \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto (-x,y) \end{aligned} \quad \text{and} \quad \begin{aligned} g\colon \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto (x,y-3). \end{aligned}$$

Give a geometric interpretation of f and g, and hence of $g \circ f$.

Section 3

Exercise 3.1

Use the exact values of sine, cosine and tangent of the angles $\frac{3}{4}\pi$ and $\frac{1}{3}\pi$ to determine exact values for the sine, cosine and tangent of $\frac{13}{12}\pi$.

Exercise 3.2

(a) Use the double-angle formula for $\tan(2\theta)$ with $\theta = \frac{1}{8}\pi$ to determine the exact value of $\tan\left(\frac{1}{8}\pi\right)$.

(b) In Section 3 of Chapter A3, it is shown that

$$\sin\left(\tfrac{1}{8}\pi\right) = \tfrac{1}{2}\sqrt{2-\sqrt{2}},$$
$$\cos\left(\tfrac{1}{8}\pi\right) = \tfrac{1}{2}\sqrt{2+\sqrt{2}},$$

from which it follows that

$$\tan\left(\tfrac{1}{8}\pi\right) = \frac{\sin\left(\frac{1}{8}\pi\right)}{\cos\left(\frac{1}{8}\pi\right)} = \frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}}.$$

Your answer to part (a) will probably look very different from this, so explain why the answers are actually the same.

Exercise 3.3

Find $\sin(2\theta)$, where θ is the angle in the interval $(-\frac{1}{2}\pi, 0)$ for which $\cos\theta = \frac{1}{3}$.

Exercise 3.4

Find $\cos\theta$ and $\sin\theta$, where θ is the angle in the interval $(\frac{1}{2}\pi, \pi)$ for which $\sec(2\theta) = 5$.

Exercise 3.5

(a) By using the sum and difference formulas for $\sin(\phi+\theta)$ and $\sin(\phi-\theta)$ with $\phi = \frac{1}{2}(A+B)$ and $\theta = \frac{1}{2}(A-B)$, derive the formula

$$\sin A + \sin B = 2\sin(\tfrac{1}{2}(A+B))\cos(\tfrac{1}{2}(A-B)).$$

(b) Adapt the method of part (a) to derive the formula

$$\sin A - \sin B = 2\cos(\tfrac{1}{2}(A+B))\sin(\tfrac{1}{2}(A-B)).$$

Exercise 3.6

By writing $\cos(3\theta)$ as $\cos(2\theta+\theta)$ and using appropriate formulas from Section 3 of Chapter A3, derive the formula

$$\cos(3\theta) = 4\cos^3\theta - 3\cos\theta.$$

Section 4

Exercise 4.1

Use the strategy of Section 4 of Chapter A3 to sketch the quadratic curve L with each of the following equations.

(a) $x^2 + 4xy - y^2 - 4\sqrt{5} = 0$

(b) $8x^2 - 4\sqrt{3}xy + 4y^2 - 2 = 0$

(c) $3x^2 + 2xy + 3y^2 - 2 = 0$

Solutions for Chapter A1

Solution 1.1

(a) On multiplying numerator and denominator by $1+\sqrt{5}$, we obtain

$$\frac{1}{1-\sqrt{5}} = \frac{1+\sqrt{5}}{(1-\sqrt{5})(1+\sqrt{5})} = \frac{1+\sqrt{5}}{1-5} = -\tfrac{1}{4}(1+\sqrt{5}).$$

(b) On multiplying numerator and denominator by $\sqrt{17}-\sqrt{2}$, we obtain

$$\frac{\sqrt{2}}{\sqrt{17}+\sqrt{2}} = \frac{\sqrt{2}(\sqrt{17}-\sqrt{2})}{(\sqrt{17}+\sqrt{2})(\sqrt{17}-\sqrt{2})} = \frac{\sqrt{34}-2}{17-2} = \tfrac{1}{15}(\sqrt{34}-2).$$

Solution 1.2

(a) In this case,

$$\alpha+\beta = -\frac{5}{1} = -5 \quad \text{and} \quad \alpha\beta = \frac{-7}{1} = -7.$$

Therefore

$$\alpha^3+\beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta) = (-5)^3 - 3(-7)(-5) = -230.$$

(b) In this case,

$$\alpha+\beta = -\frac{-9}{5} = \frac{9}{5} \quad \text{and} \quad \alpha\beta = \frac{1}{5}.$$

Therefore

$$\alpha^3+\beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta) = \left(\frac{9}{5}\right)^3 - 3\left(\frac{1}{5}\right)\left(\frac{9}{5}\right) = \frac{594}{125}.$$

Solution 3.1

(a) The auxiliary equation is

$$r^2-5r-6 = (r-6)(r+1) = 0.$$

This has solutions $r=6$ and $r=-1$.

Thus the general solution is

$$u_n = A6^n + B(-1)^n,$$

where A and B are constants.

To find A and B, we use the initial terms:

$$u_0 = 3 \quad \text{gives} \quad A+B=3,$$
$$u_1 = 7 \quad \text{gives} \quad 6A-B=7.$$

Hence $A=\frac{10}{7}$, $B=\frac{11}{7}$, so

$$u_n = \tfrac{10}{7}6^n + \tfrac{11}{7}(-1)^n.$$

(b) The auxiliary equation is

$$r^2+8r+16 = (r+4)^2 = 0.$$

This has the (repeated) solution $r=-4$.

Thus the general solution is

$$u_n = (A+Bn)(-4)^n,$$

where A and B are constants.

To find A and B, we use the initial terms:

$$u_0 = 5 \quad \text{gives} \quad A=5,$$
$$u_1 = 20 \quad \text{gives} \quad (A+B)(-4)=20.$$

Hence $A=5$, $B=-10$, so

$$u_n = (5-10n)(-4)^n.$$

(c) The auxiliary equation is

$$r^2-1.8r+0.77 = 0.$$

By the formula,

$$r = \frac{-(-1.8) \pm \sqrt{(-1.8)^2 - 4\times 1\times 0.77}}{2\times 1} = \frac{1.8 \pm \sqrt{0.16}}{2} = \frac{1.8 \pm 0.4}{2},$$

so the solutions are $r=1.1$ and $r=0.7$.

Thus the general solution is

$$u_n = A(1.1)^n + B(0.7)^n,$$

where A and B are constants.

To find A and B, we use the initial terms:

$$u_0 = 4 \quad \text{gives} \quad A+B=4,$$
$$u_1 = 6 \quad \text{gives} \quad 1.1A+0.7B=6.$$

Hence $A=8$, $B=-4$, so

$$u_n = 8(1.1)^n - 4(0.7)^n.$$

Solution 5.1

(a) First checking the initial terms, we have

$$u_0 = 7^0 - (-5)^0 = 1-1 = 0,$$
$$u_1 = 7^1 - (-5)^1 = 7-(-5) = 12,$$

as required.

To show that the recurrence relation

$$u_{n+2} = 2u_{n+1} + 35u_n$$

is satisfied, we substitute for u_{n+2} in the LHS and for u_{n+1} and u_n in the RHS and show that

the two sides are equal. We have

$$\begin{aligned}\text{LHS} &= u_{n+2} = 7^{n+2} - (-5)^{n+2}\\ &= 49 \times 7^n - 25(-5)^n;\end{aligned}$$

$$\begin{aligned}\text{RHS} &= 2u_{n+1} + 35u_n\\ &= 2(7^{n+1} - (-5)^{n+1}) + 35(7^n - (-5)^n)\\ &= 2(7 \times 7^n + 5 \times (-5)^n) + 35(7^n - (-5)^n)\\ &= 49 \times 7^n - 25(-5)^n.\end{aligned}$$

So LHS = RHS.

(b) We substitute for u_{n-1}, u_{n+1} and u_n in the LHS, $u_{n-1}u_{n+1} - u_n^2$, and show that the result simplifies to $-144(-35)^{n-1}$, which is the RHS. We have, for $n = 1, 2, 3, \ldots$,

$$\begin{aligned}u_{n-1}u_{n+1} &= \left(7^{n-1} - (-5)^{n-1}\right)\\ &\quad \times \left(7^{n+1} - (-5)^{n+1}\right)\\ &= 7^{2n} - 7^{n-1}(-5)^{n+1}\\ &\quad - (-5)^{n-1}7^{n+1} + (-5)^{2n},\end{aligned}$$

and

$$\begin{aligned}u_n^2 &= (7^n - (-5)^n)^2\\ &= 7^{2n} - 2(7^n(-5)^n) + (-5)^{2n}.\end{aligned}$$

Therefore, after cancelling,

$$\begin{aligned}\text{LHS} &= 7^{n-1}(-5)^{n-1}\\ &\quad \times \left(-(-5)^2 - 7^2 + 2 \times 7 \times (-5)\right)\\ &= (-35)^{n-1}(-25 - 49 - 70)\\ &= -144(-35)^{n-1} = \text{RHS},\end{aligned}$$

as required.

Solution 5.2

(a) We use the closed form $u_n = \frac{10}{7}6^n + \frac{11}{7}(-1)^n$, from Exercise 3.1(a). We substitute for u_n in the LHS, $u_{n-1}u_{n+1} - u_n^2$, and show that the result simplifies to $110(-6)^{n-1}$, which is the RHS. We have, for $n = 1, 2, 3, \ldots$,

$$\begin{aligned}u_{n-1}u_{n+1} &= \left(\tfrac{10}{7}6^{n-1} + \tfrac{11}{7}(-1)^{n-1}\right)\\ &\quad \times \left(\tfrac{10}{7}6^{n+1} + \tfrac{11}{7}(-1)^{n+1}\right)\\ &= \left(\tfrac{10}{7}\right)^2 6^{2n}\\ &\quad + \left(\tfrac{10}{7}\right)\left(\tfrac{11}{7}\right)6^{n-1}(-1)^{n+1}\\ &\quad + \left(\tfrac{11}{7}\right)\left(\tfrac{10}{7}\right)(-1)^{n-1}6^{n+1}\\ &\quad + \left(\tfrac{11}{7}\right)^2(-1)^{2n},\end{aligned}$$

and

$$\begin{aligned}u_n^2 &= \left(\tfrac{10}{7}6^n + \tfrac{11}{7}(-1)^n\right)^2\\ &= \left(\tfrac{10}{7}\right)^2 6^{2n}\\ &\quad + 2\left(\tfrac{10}{7}\right)\left(\tfrac{11}{7}\right)6^n(-1)^n\\ &\quad + \left(\tfrac{11}{7}\right)^2(-1)^{2n}.\end{aligned}$$

Therefore, after cancelling,

$$\begin{aligned}\text{LHS} &= \left(\tfrac{10}{7}\right)\left(\tfrac{11}{7}\right)(-1)^{n-1}6^{n-1}\\ &\quad \times \left((-1)^2 + 6^2 - 2(-1)6\right)\\ &= \left(\tfrac{110}{49}\right)(-1)^{n-1}6^{n-1}(1 + 36 + 12),\\ &= \left(\tfrac{110}{49}\right)(-1)^{n-1}6^{n-1} \times 49,\\ &= 10 \times 11 \times (-1)^{n-1}6^{n-1}\\ &= 110(-6)^{n-1} = \text{RHS},\end{aligned}$$

as required.

(b) We use the closed form $u_n = (5 - 10n)(-4)^n$, from Exercise 3.1(b). We substitute for u_n in the LHS, $u_{n-1}u_{n+1} - u_n^2$, and show that the result simplifies to -100×16^n, which is the RHS. We have, for $n = 1, 2, 3, \ldots$,

$$\begin{aligned}u_{n-1}u_{n+1} &= \left(5 - 10(n-1)\right)(-4)^{n-1}\\ &\quad \times \left(5 - 10(n+1)\right)(-4)^{n+1}\\ &= (15 - 10n)(-5 - 10n)(-4)^{2n}\\ &= (-75 - 100n + 100n^2)(-4)^{2n},\end{aligned}$$

and

$$\begin{aligned}u_n^2 &= (5 - 10n)^2(-4)^{2n}\\ &= (25 - 100n + 100n^2)(-4)^{2n}.\end{aligned}$$

Therefore, after cancelling,

$$\begin{aligned}\text{LHS} &= -100(-4)^{2n}\\ &= -100 \times 16^n = \text{RHS},\end{aligned}$$

as required.

Solution 5.3

(a) Putting the LHS of the required identity over a common denominator gives:

$$\begin{aligned}\frac{1}{F_{n+1}} - \frac{1}{F_{n+2}} &= \frac{F_{n+2} - F_{n+1}}{F_{n+1}F_{n+2}}\\ &= \frac{F_n}{F_{n+1}F_{n+2}}\end{aligned}$$

(as $F_{n+2} = F_{n+1} + F_n$)

which is the RHS of the identity.

(b) The identity in part (a) gives the $n + 1$ equations

$$\begin{aligned}\frac{F_0}{F_1F_2} &= \frac{1}{F_1} - \frac{1}{F_2}\\ \frac{F_1}{F_2F_3} &= \frac{1}{F_2} - \frac{1}{F_3}\\ \frac{F_2}{F_3F_4} &= \frac{1}{F_3} - \frac{1}{F_4}\\ &\vdots\\ \frac{F_n}{F_{n+1}F_{n+2}} &= \frac{1}{F_{n+1}} - \frac{1}{F_{n+2}}.\end{aligned}$$

If we add these equations, then

◇ on the LHS, we obtain the sum

$$\frac{F_0}{F_1F_2}+\frac{F_1}{F_2F_3}+\cdots+\frac{F_n}{F_{n+1}F_{n+2}};$$

◇ on the RHS, we obtain

$\dfrac{1}{F_1}-\dfrac{1}{F_{n+2}}$, by telescoping cancellation.

Since $F_1 = 1$,

$$\frac{F_0}{F_1F_2}+\frac{F_1}{F_2F_3}+\cdots+\frac{F_n}{F_{n+1}F_{n+2}}=1-\frac{1}{F_{n+2}},$$

for $n = 0, 1, 2, \ldots$, as required.

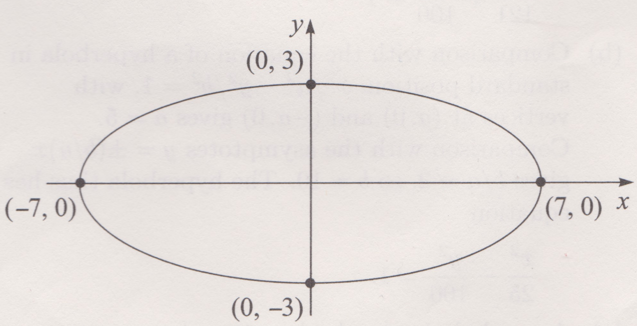
Solutions for Chapter A2

Solution 2.1

(a) On dividing $9x^2 + 49y^2 = 441$ throughout by 441 (to give 1 on the RHS), we obtain

$$\frac{x^2}{49}+\frac{y^2}{9}=1;\quad \text{that is,}\quad \frac{x^2}{7^2}+\frac{y^2}{3^2}=1.$$

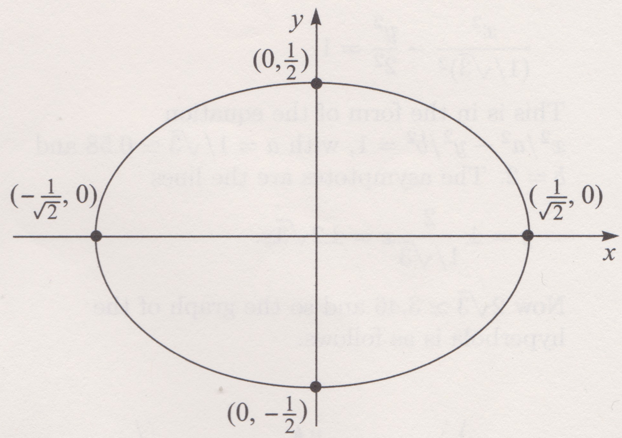
This is in the form of the equation $x^2/a^2 + y^2/b^2 = 1$, with $a = 7$ and $b = 3$. The graph of the ellipse is as follows.



(b) First rearrange the equation so that the constant term appears on the RHS of the equation. On dividing $6x^2 + 12y^2 = 3$ throughout by 3 (to give 1 on the RHS), we obtain

$$2x^2+4y^2=1;\quad \text{that is,}\quad \frac{x^2}{(1/\sqrt{2})^2}+\frac{y^2}{(1/2)^2}=1.$$

This is in the form of the equation $x^2/a^2 + y^2/b^2 = 1$, with $a = 1/\sqrt{2} \simeq 0.71$ and $b = 1/2$. The graph of the ellipse is as follows.

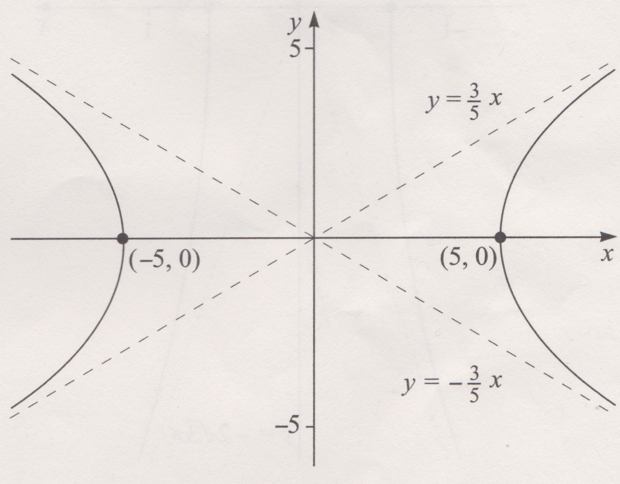


Solution 2.2

(a) On dividing $9x^2 - 25y^2 = 225$ throughout by 225 (to give 1 on the RHS), we obtain

$$\frac{x^2}{25}-\frac{y^2}{9}=1;\quad \text{that is,}\quad \frac{x^2}{5^2}-\frac{y^2}{3^2}=1.$$

This is in the form of the equation $x^2/a^2 - y^2/b^2 = 1$, with $a = 5$ and $b = 3$. The asymptotes are the lines $y = \pm\frac{3}{5}x$. The graph of the hyperbola is as follows.



(b) First rearrange the equation $3y^2 + 12 - 36x^2 = 0$ so that the constant term appears as a positive number on the RHS of the equation: $36x^2 - 3y^2 = 12$. Next rearrange the equation so that 1 appears on the RHS of the equation:

$$3x^2-\frac{y^2}{4}=1;$$

that is,

$$\frac{x^2}{1/3}-\frac{y^2}{4}=1,$$

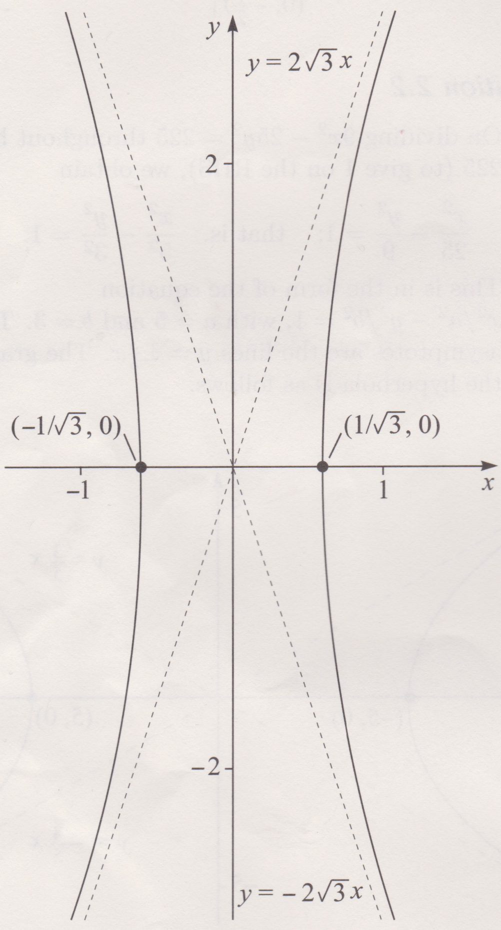
or

$$\frac{x^2}{(1/\sqrt{3})^2} - \frac{y^2}{2^2} = 1.$$

This is in the form of the equation $x^2/a^2 - y^2/b^2 = 1$, with $a = 1/\sqrt{3} \simeq 0.58$ and $b = 2$. The asymptotes are the lines

$$y = \pm\frac{2}{1/\sqrt{3}}x = \pm 2\sqrt{3}x.$$

Now $2\sqrt{3} \simeq 3.46$ and so the graph of the hyperbola is as follows.

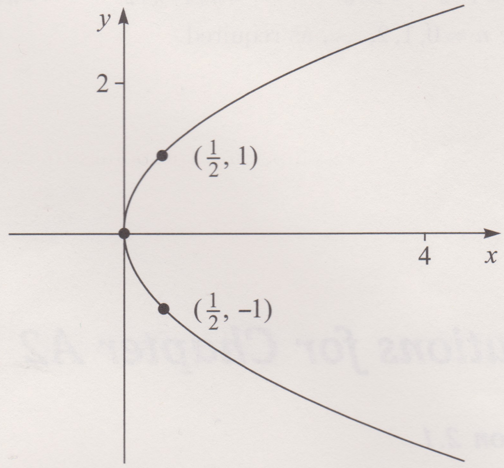


Solution 2.3

To write the equation $16x - 8y^2 = 0$ in the form of equation $y^2 = 4ax$, we first rearrange the equation to isolate y^2 and obtain $y^2 = 2x$. Then we replace $2x$ by $(4 \times \frac{1}{2})x$. Now

$$y^2 = (4 \times \tfrac{1}{2})x$$

is in the form of the equation $y^2 = 4ax$ with $a = \frac{1}{2}$. The graph of the parabola is as follows.



Solution 2.4

(a) Comparison with the equation of an ellipse in standard position, $x^2/a^2 + y^2/b^2 = 1$, with vertices at $(a, 0)$, $(-a, 0)$, $(0, b)$, $(0, -b)$ gives $a = 11$ and $b = 10$, so the ellipse has equation

$$\frac{x^2}{121} + \frac{y^2}{100} = 1.$$

(b) Comparison with the equation of a hyperbola in standard position, $x^2/a^2 - y^2/b^2 = 1$, with vertices at $(a, 0)$ and $(-a, 0)$ gives $a = 5$. Comparison with the asymptotes $y = \pm(b/a)x$ gives $b/a = 2$, so $b = 10$. The hyperbola thus has equation

$$\frac{x^2}{25} - \frac{y^2}{100} = 1.$$

(c) A parabola in standard position has equation $y^2 = 4ax$. If the parabola includes the point $(1, -6)$, then a satisfies the equation $(-6)^2 = 4a \times 1$, so $a = 9$. The equation of the parabola is thus $y^2 = 36x$.

Solution 3.1

(a) (i) The equation $9x^2 + 49y^2 = 441$ can be rearranged as

$$\frac{x^2}{7^2} + \frac{y^2}{3^2} = 1,$$

so $a = 7$ and $b = 3$. The eccentricity of this ellipse is

$$\begin{aligned} e = \sqrt{1 - b^2/a^2} &= \sqrt{1 - 9/49} \\ &= \sqrt{40/49} \\ &= \tfrac{2}{7}\sqrt{10} \simeq 0.90. \end{aligned}$$

Thus

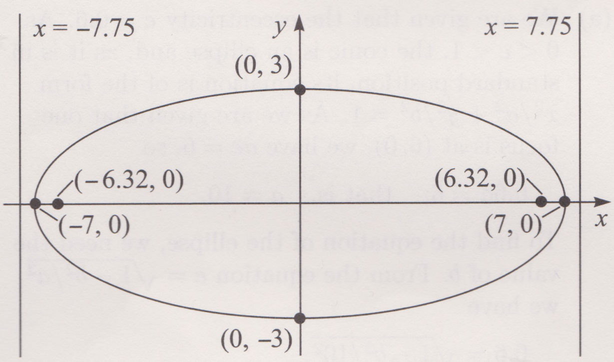
$$ae = 7 \times \tfrac{2}{7}\sqrt{10} = 2\sqrt{10} \simeq 6.32,$$

so the foci of this ellipse are the points $(6.32, 0)$ and $(-6.32, 0)$. Also

$$a/e = 7/(\tfrac{2}{7}\sqrt{10}) = 49/(2\sqrt{10}) \simeq 7.75,$$

so the corresponding directrices of this ellipse are the lines $x = 7.75$ and $x = -7.75$.

The foci and directrices are as follows.



(ii) The equation $6x^2 + 12y^2 - 3 = 0$ can be rearranged as

$$\frac{x^2}{(1/\sqrt{2})^2} + \frac{y^2}{(1/2)^2} = 1,$$

so $a = \frac{1}{\sqrt{2}} \simeq 0.71$ and $b = \frac{1}{2}$. The eccentricity of this ellipse is

$$\begin{aligned} e = \sqrt{1 - b^2/a^2} &= \sqrt{1 - \tfrac{1}{4}/\tfrac{1}{2}} \\ &= \sqrt{\tfrac{1}{2}} = \tfrac{1}{\sqrt{2}} \simeq 0.71. \end{aligned}$$

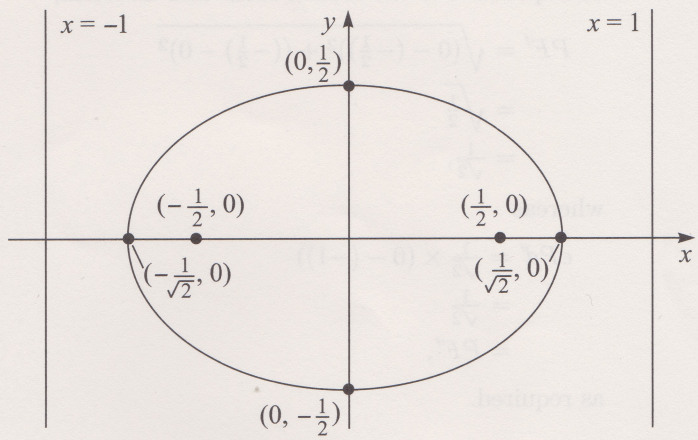
Thus

$$ae = \tfrac{1}{\sqrt{2}} \times \tfrac{1}{\sqrt{2}} = \tfrac{1}{2},$$

so the foci of this ellipse are the points $(\frac{1}{2}, 0)$ and $(-\frac{1}{2}, 0)$. Also

$$a/e = \tfrac{1}{\sqrt{2}} / \tfrac{1}{\sqrt{2}} = 1,$$

so the corresponding directrices of this ellipse are the lines $x = 1$ and $x = -1$.



(b) From the solution to part (a)(ii), the eccentricity of the ellipse is $\frac{1}{\sqrt{2}}$, the focus $F(\frac{1}{2}, 0)$ corresponds to the directrix d with equation $x = 1$, and the focus $F'(-\frac{1}{2}, 0)$ corresponds to the directrix d' with equation $x = -1$.

For the point $P(\frac{1}{\sqrt{2}}, 0)$,

$$PF = \tfrac{1}{\sqrt{2}} - \tfrac{1}{2},$$

whereas

$$\begin{aligned} ePd &= \tfrac{1}{\sqrt{2}} \times (1 - \tfrac{1}{\sqrt{2}}) \\ &= \tfrac{1}{\sqrt{2}} - \tfrac{1}{2} \\ &= PF, \end{aligned}$$

as required. For the second focus and directrix

$$\begin{aligned} PF' &= \tfrac{1}{\sqrt{2}} - (-\tfrac{1}{2}) \\ &= \tfrac{1}{2} + \tfrac{1}{\sqrt{2}}, \end{aligned}$$

whereas

$$\begin{aligned} ePd' &= \tfrac{1}{\sqrt{2}} \times \left(\tfrac{1}{\sqrt{2}} - (-1)\right) \\ &= \tfrac{1}{\sqrt{2}} + \tfrac{1}{2} \\ &= PF', \end{aligned}$$

as required.

For the point $P(0, -\frac{1}{2})$,

$$\begin{aligned} PF &= \sqrt{(0 - \tfrac{1}{2})^2 + ((-\tfrac{1}{2}) - 0)^2} \\ &= \sqrt{\tfrac{1}{2}} \\ &= \tfrac{1}{\sqrt{2}}, \end{aligned}$$

whereas

$$\begin{aligned} ePd &= \tfrac{1}{\sqrt{2}} \times (1 - 0) \\ &= \tfrac{1}{\sqrt{2}} \\ &= PF, \end{aligned}$$

as required. For the second focus and directrix,

$$\begin{aligned} PF' &= \sqrt{(0-(-\tfrac{1}{2}))^2+((-\tfrac{1}{2})-0)^2} \\ &= \sqrt{\tfrac{1}{2}} \\ &= \tfrac{1}{\sqrt{2}}, \end{aligned}$$

whereas

$$\begin{aligned} ePd' &= \tfrac{1}{\sqrt{2}} \times (0-(-1)) \\ &= \tfrac{1}{\sqrt{2}} \\ &= PF', \end{aligned}$$

as required.

Solution 3.2

(a) The equation $9x^2 - 25y^2 = 225$ can be rearranged as

$$\frac{x^2}{5^2} - \frac{y^2}{3^2} = 1,$$

so $a = 5$ and $b = 3$. The eccentricity of this hyperbola is

$$e = \sqrt{1 + b^2/a^2} = \sqrt{1 + 9/25} = \sqrt{34}/5.$$

Thus

$$ae = 5 \times \sqrt{34}/5 = \sqrt{34},$$

so the foci of this hyperbola are the points $(\sqrt{34}, 0)$ and $(-\sqrt{34}, 0)$. Also

$$a/e = 5/(\sqrt{34}/5) = 25/\sqrt{34},$$

so the corresponding directrices of this hyperbola are the lines $x = 25/\sqrt{34}$ and $x = -25/\sqrt{34}$.

(b) The equation $3y^2 + 12 - 36x^2 = 0$ can be rearranged as

$$\frac{x^2}{(1/\sqrt{3})^2} - \frac{y^2}{2^2} = 1,$$

so $a = 1/\sqrt{3}$ and $b = 2$. The eccentricity of this hyperbola is

$$e = \sqrt{1 + b^2/a^2} = \sqrt{1 + 4/(1/3)} = \sqrt{13}.$$

Thus

$$ae = \frac{1}{\sqrt{3}} \times \sqrt{13} = \sqrt{13/3},$$

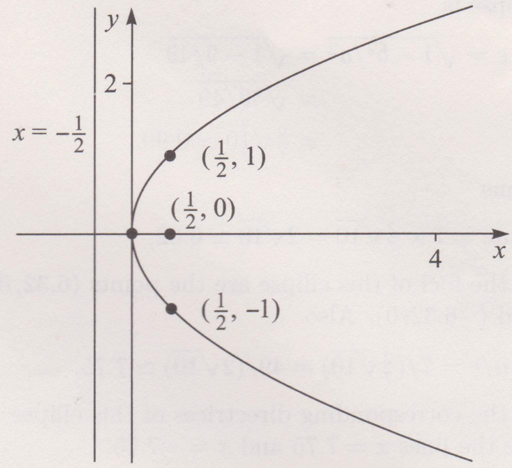
so the foci of this hyperbola are the points $(\sqrt{13/3}, 0)$ and $(-\sqrt{13/3}, 0)$. Also

$$a/e = \frac{1}{\sqrt{3}} \bigg/ \sqrt{13} = \sqrt{1/39},$$

so the corresponding directrices of this hyperbola are the lines $x = 1/\sqrt{39}$ and $x = -1/\sqrt{39}$.

Solution 3.3

The equation $16x - 8y^2 = 0$ can be rearranged as $y^2 = (4 \times \frac{1}{2})\,x$, so $a = \frac{1}{2}$. Thus the focus is the point $(\frac{1}{2}, 0)$ and the directrix is the line $x = -\frac{1}{2}$, as shown in the following figure.



Solution 3.4

(a) We are given that the eccentricity $e = 0.6$. As $0 < e < 1$, the conic is an ellipse and, as it is in standard position, its equation is of the form $x^2/a^2 + y^2/b^2 = 1$. As we are given that one focus is at $(6, 0)$, we have $ae = 6$, so

$$0.6a = 6; \quad \text{that is,} \quad a = 10.$$

To find the equation of the ellipse, we need the value of b. From the equation $e = \sqrt{1 - b^2/a^2}$, we have

$$0.6 = \sqrt{1 - b^2/10^2}.$$

Squaring each side gives

$$0.36 = 1 - \frac{b^2}{100},$$

so

$$b^2 = 100 - 100 \times 0.36 = 100 - 36 = 64;$$

that is, $b = 8$ (taking the positive square root, as the convention for standard form is to take $b > 0$). The equation of the ellipse is thus

$$\frac{x^2}{10^2} + \frac{y^2}{8^2} = 1; \quad \text{that is} \quad \frac{x^2}{100} + \frac{y^2}{64} = 1.$$

(Strictly speaking, we didn't need to work out the value of b, as to answer the question we only needed the value of b^2.)

(b) We are given that the eccentricity $e = 6$. As $e > 1$, the conic is a hyperbola and, as it is in standard position, its equation is of the form $x^2/a^2 - y^2/b^2 = 1$. As we are given that one directrix is $x = \frac{1}{3}$, we have $a/e = \frac{1}{3}$, so

$$\frac{a}{6} = \frac{1}{3}; \quad \text{that is,} \quad a = 2.$$

To find the equation of the hyperbola, we need the value of b. From the equation $e = \sqrt{1 + b^2/a^2}$, we have

$$6 = \sqrt{1 + b^2/2^2}.$$

Squaring each side gives

$$36 = 1 + \frac{b^2}{4},$$

so

$$b^2 = 4 \times 36 - 4 = 144 - 4 = 140;$$

that is, $b = \sqrt{140}$ (taking the positive square root, as the convention for standard form is to take $b > 0$). The equation of the hyperbola is thus

$$\frac{x^2}{2^2} - \frac{y^2}{(\sqrt{140})^2} = 1; \quad \text{that is} \quad \frac{x^2}{4} - \frac{y^2}{140} = 1.$$

(c) Since the focus $(5, 0)$ and directrix $x = -5$ are equidistant from the origin, this conic in standard position must be a parabola with $a = 5$. So the equation of the conic is $y^2 = 4 \times 5x$; that is $y^2 = 20x$.

(d) Since the directrix is further from the origin than the focus, this conic in standard position must be an ellipse. The constants a and e must satisfy the equations

$$ae = 2 \quad \text{and} \quad \frac{a}{e} = 8.$$

Eliminating e from these equations gives

$$a^2 = 16,$$

so $a = 4$ (since $a > 0$). Substituting this value of a into the equation $ae = 2$ gives $e = \frac{1}{2}$. (Note that $0 < e < 1$, as expected for an ellipse.)

To find the equation of the ellipse, we need the value of b. From the equation $e = \sqrt{1 - b^2/a^2}$,

$$\frac{1}{2} = \sqrt{1 - b^2/4^2}.$$

Squaring each side gives

$$\frac{1}{4} = 1 - \frac{b^2}{16},$$

so

$$b^2 = 16 - 4 = 12;$$

that is, $b = \sqrt{12} = 2\sqrt{3}$. The equation of the ellipse is thus

$$\frac{x^2}{4^2} + \frac{y^2}{(\sqrt{12})^2} = 1; \quad \text{that is} \quad \frac{x^2}{16} + \frac{y^2}{12} = 1.$$

Solution 3.5

(a)
$$\begin{aligned} PF &= \sqrt{(x - 18)^2 + y^2} \\ &= \sqrt{x^2 - 36x + 324 + y^2}. \end{aligned}$$

The point P could lie on the line d or either side of it. If P lies on the line, its distance Pd to d is 0. If P lies to the right of d, $Pd = x - 2$; and if it lies to the left of d, $Pd = 2 - x = -(x - 2)$.

(b) The equation $PF = 3Pd$ gives

$$\sqrt{x^2 - 36x + 324 + y^2} = \pm 3(x - 2),$$

with the sign of the RHS depending on which side of the line d the point P is situated. Regardless of the sign, squaring both sides of the equation gives

$$\begin{aligned} x^2 - 36x + 324 + y^2 &= 9(x - 2)^2, \\ &= 9x^2 - 36x + 36. \end{aligned}$$

By cancelling and collecting like terms, we obtain

$$8x^2 - y^2 = 288,$$

and on dividing throughout by 288, we obtain

$$\frac{x^2}{36} - \frac{y^2}{288} = 1,$$

which is the equation of a hyperbola in standard position with $a = 6$ and $b = 12\sqrt{2}$.

Solution 4.1

(a) On collecting x and y terms, we obtain

$$36(x^2 - 2x) - 25(y^2 + 4y) = 964.$$

On completing the squares, we then obtain

$$36((x - 1)^2 - 1) - 25((y + 2)^2 - 4) = 964;$$

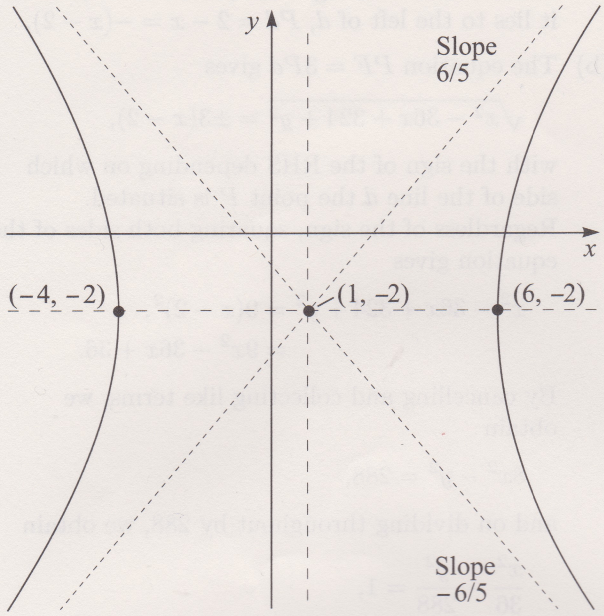
that is,

$$36(x - 1)^2 - 25(y + 2)^2 = 900,$$

or

$$\frac{(x - 1)^2}{25} - \frac{(y + 2)^2}{36} = 1.$$

This is the hyperbola in standard position with $a = 5$, $b = 6$, translated to have centre $(1, -2)$. The hyperbola is as follows.



(b) On collecting x and y terms, we obtain

$$4(x^2 + x) + 8(y^2 + 6y) = -41.$$

On completing the squares, we then obtain

$$4((x + \tfrac{1}{2})^2 - \tfrac{1}{4}) + 8((y + 3)^2 - 9) = -41;$$

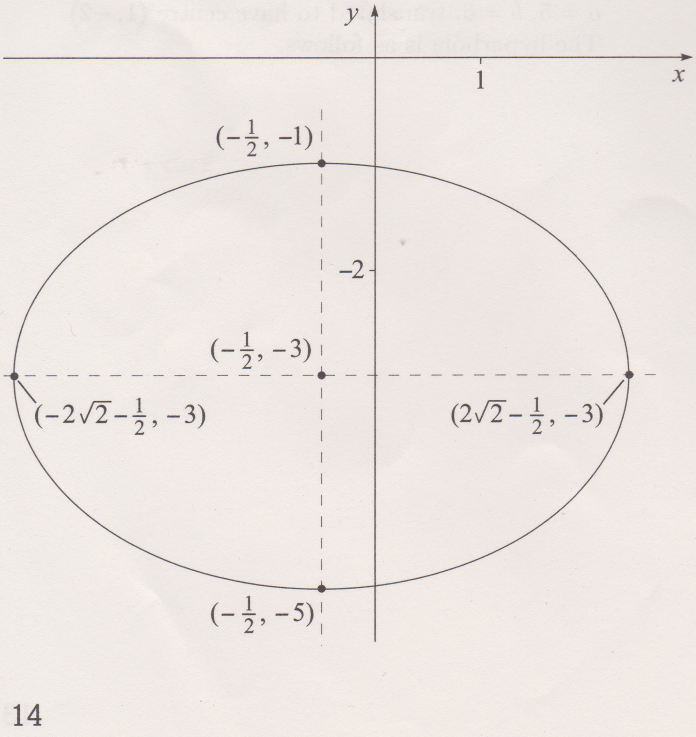
that is,

$$4(x + \tfrac{1}{2})^2 + 8(y + 3)^2 = 32,$$

or

$$\frac{(x + \frac{1}{2})^2}{8} + \frac{(y + 3)^2}{4} = 1.$$

This is the ellipse in standard position with $a = 2\sqrt{2} \simeq 2.83$, $b = 2$, translated to have centre $(-\frac{1}{2}, -3)$. The ellipse is as follows.



(c) On collecting y terms, we obtain

$$(y^2 + 8y) - 12x + 40 = 0.$$

On completing the square, we then obtain

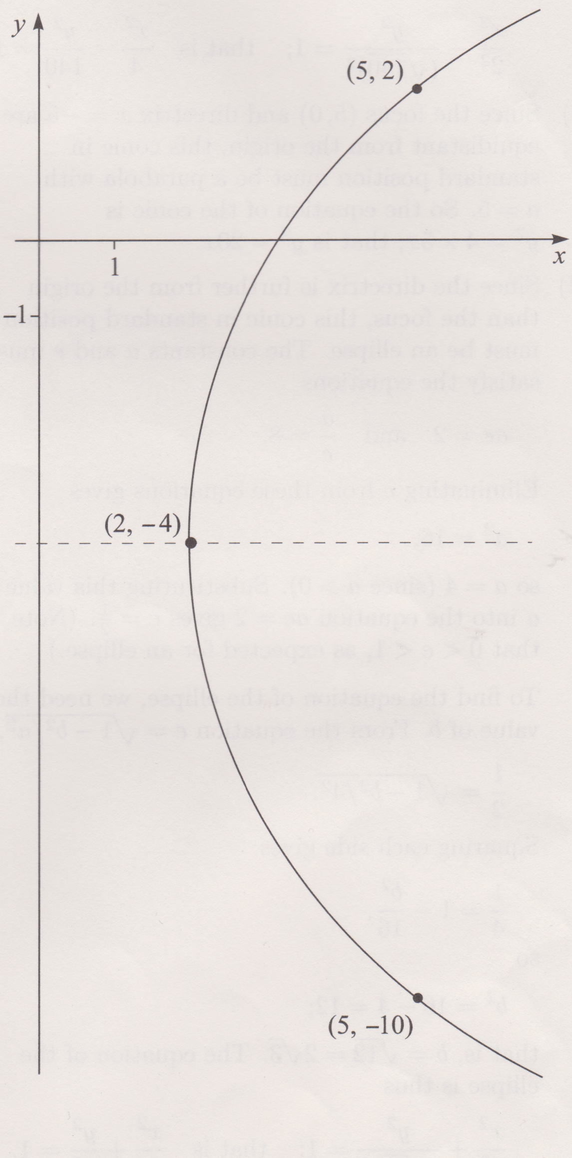
$$(y + 4)^2 - 16 - 12x + 40 = 0;$$

that is,

$$\begin{aligned}(y + 4)^2 &= 12x - 24\\ &= 12(x - 2) = (4 \times 3)(x - 2).\end{aligned}$$

This is the parabola in standard position with $a = 3$, translated to have vertex $(2, -4)$.

The parabola is sketched below.



Solution 5.1

(a) This equation can be rearranged as

$$y^2 = \tfrac{1}{2}x = (4 \times \tfrac{1}{8})x,$$

so this is a parabola in standard position with $a = \frac{1}{8}$. The standard parametrisation is

$$x = \tfrac{1}{8}t^2, \quad y = \tfrac{1}{4}t.$$

(b) This equation can be rearranged as

$$\frac{x^2}{9} + \frac{y^2}{3/4} = 1,$$

so this is an ellipse in standard position with $a = 3$, $b = \frac{1}{2}\sqrt{3}$. The standard parametrisation is

$$x = 3\cos t, \quad y = \tfrac{1}{2}\sqrt{3}\sin t \quad (0 \le t \le 2\pi).$$

(c) This equation can be rearranged as

$$\frac{x^2}{1/7} - \frac{y^2}{1/4} = 1,$$

so this is a hyperbola in standard position with $a = 1/\sqrt{7}$, $b = \frac{1}{2}$. The standard parametrisation is

$$x = \tfrac{1}{\sqrt{7}}\sec t, \quad y = \tfrac{1}{2}\tan t \quad (-\tfrac{1}{2}\pi < t < \tfrac{1}{2}\pi, \tfrac{1}{2}\pi < t < \tfrac{3}{2}\pi).$$

Solution 5.2

(a) From Exercise 4.1(a), we know that the equation $36x^2 - 25y^2 - 72x - 100y - 964 = 0$ can be rearranged as

$$\frac{(x-1)^2}{25} - \frac{(y+2)^2}{36} = 1.$$

This is the hyperbola in standard position with $a = 5$, $b = 6$, translated to have centre $(1, -2)$, so a parametrisation is

$$x = 1 + 5\sec t, \quad y = -2 + 6\tan t \quad (-\tfrac{1}{2}\pi < t < \tfrac{1}{2}\pi, \tfrac{1}{2}\pi < t < \tfrac{3}{2}\pi).$$

(b) From Exercise 4.1(b), we know that the equation $4x^2 + 8y^2 + 4x + 48y + 41 = 0$ can be rearranged as

$$\frac{(x+\frac{1}{2})^2}{8} + \frac{(y+3)^2}{4} = 1.$$

This is the ellipse in standard position with $a = 2\sqrt{2}$, $b = 2$, translated to have centre $(-\frac{1}{2}, -3)$, so a parametrisation is

$$x = -\tfrac{1}{2} + 2\sqrt{2}\cos t, \quad y = -3 + 2\sin t \quad (0 \le t \le 2\pi).$$

(c) From Exercise 4.1(c), we know that the equation $y^2 - 12x + 8y + 40 = 0$ can be rearranged as

$$(y+4)^2 = (4 \times 3)(x - 2).$$

This is the parabola in standard position with $a = 3$, translated to have vertex $(2, -4)$, so a parametrisation is

$$x = 2 + 3t^2, \quad y = -4 + 6t.$$

Solution 5.3

(a) The equations parametrise an ellipse in standard position with $a = 4$ and $b = \sqrt{7}$. The eccentricity of this ellipse is

$$\begin{aligned} e &= \sqrt{1 - b^2/a^2} \\ &= \sqrt{1 - 7/16} \\ &= \sqrt{9/16} = \tfrac{3}{4}. \end{aligned}$$

(b) The equations parametrise a parabola in standard position with $a = 5$. The eccentricity of any parabola is 1.

(c) The equations parametrise a hyperbola congruent to the one in standard position parametrised by

$$x = 4\sec t, \quad y = 3\tan t \quad (-\tfrac{1}{2}\pi < t < \tfrac{1}{2}\pi, \tfrac{1}{2}\pi < t < \tfrac{3}{2}\pi).$$

The eccentricty of both hyperbolas is

$$\begin{aligned} e &= \sqrt{1 + b^2/a^2} \\ &= \sqrt{1 + 9/16} \\ &= \sqrt{25/16} = \tfrac{5}{4}. \end{aligned}$$

Solutions for Chapter A3

Solution 1.1

(a) Note that the curves are congruent, and that the centres of K and L are at $(0,0)$ and $(5,-1)$, respectively. To map K onto L, we need to translate the points of K five units to the right and one unit down. Thus t is the function

$$\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x + 5, y - 1). \end{aligned}$$

(b) To map L onto K, we need to translate the points of L five units to the left and one unit up. Thus t is the function

$$\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x - 5, y + 1). \end{aligned}$$

Solution 1.2

(a) The circle has radius 3 and centre $(-4, 1)$, so it has parametric equations

$$x = -4 + 3\cos t, \quad y = 1 + 3\sin t \quad (0 \le t \le 2\pi).$$

Thus the circle has parametrisation function

$$\begin{aligned} p : [0, 2\pi] &\longrightarrow \mathbb{R}^2 \\ t &\longmapsto (-4 + 3\cos t, 1 + 3\sin t). \end{aligned}$$

(There are many other correct parametrisation functions.)

(b) This is an ellipse in standard position

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

with $a = \sqrt{7}$ and $b = \sqrt{3}$, so it has parametric equations

$$x = \sqrt{7}\cos t, \quad y = \sqrt{3}\sin t \quad (0 \le t \le 2\pi).$$

A suitable parametrisation function for L is therefore

$$\begin{aligned} p : [0, 2\pi] &\longrightarrow \mathbb{R}^2 \\ t &\longmapsto (\sqrt{7}\cos t, \sqrt{3}\sin t). \end{aligned}$$

(c) This is a hyperbola in standard position

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

with $a = 2$ and $b = \sqrt{5}$, so it has parametric equations

$$x = 2\sec t, \quad y = \sqrt{5}\tan t$$
$$\left(-\tfrac{1}{2}\pi < t < \tfrac{1}{2}\pi, \tfrac{1}{2}\pi < t < \tfrac{3}{2}\pi\right).$$

A parametrisation function for the left branch is thus

$$\begin{aligned} p : \left(\tfrac{1}{2}\pi, \tfrac{3}{2}\pi\right) &\longrightarrow \mathbb{R}^2 \\ t &\longmapsto (2\sec t, \sqrt{5}\tan t). \end{aligned}$$

Solution 1.3

(a) Let us first rewrite the equation of the curve in the form

$$(x^2 + 2x) + 2(y^2 - 4y) = 7.$$

On completing the squares, we then obtain

$$((x+1)^2 - 1) + 2((y-2)^2 - 4) = 7;$$

that is,

$$(x+1)^2 + 2(y-2)^2 = 16,$$

or

$$\frac{(x+1)^2}{16} + \frac{(y-2)^2}{8} = 1.$$

It follows that the quadratic curve L is an ellipse with centre $(-1, 2)$ that can be moved into standard position by translating one unit to the right and two units down. The translation function that achieves this is

$$\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x + 1, y - 2). \end{aligned}$$

The image of L under t is the ellipse K with equation

$$\frac{x^2}{16} + \frac{y^2}{8} = 1.$$

(b) To map the image ellipse K back onto the quadratic curve L we need to translate it one unit to the left and two units up. The translation function that achieves this is

$$\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x - 1, y + 2). \end{aligned}$$

(c) The ellipse $x^2/16 + y^2/8 = 1$ has parametric equations

$$x = 4\cos t, \quad y = 2\sqrt{2}\sin t \quad (0 \le t \le 2\pi).$$

A suitable parametrisation function for L is therefore

$$\begin{aligned} p : [0, 2\pi] &\longrightarrow \mathbb{R}^2 \\ t &\longmapsto (-1 + 4\cos t, 2 + 2\sqrt{2}\sin t). \end{aligned}$$

Solution 2.1

(a) The inverse of $t_{-7,9}$ is $t_{7,-9}$.

(b) The composite $t_{0,4} \circ t_{-3,1}$ is $t_{0-3,4+1} = t_{-3,5}$.

Solution 2.2

(a) The inverse of $r_{-2\pi/7}$ is $r_{-(-2\pi/7)}$, that is, $r_{2\pi/7}$. (The angle $2\pi/7$ lies in the interval $(-\pi, \pi]$.)

(b) The composite $r_{5\pi/6} \circ r_{2\pi/3}$ is $r_{3\pi/2} = r_{-\pi/2}$. (The angle $3\pi/2 - 2\pi = -\pi/2$ lies in the interval $(-\pi, \pi]$.)

Solution 2.3

(a) The required translation is

$$\begin{aligned} t_{9,-4} : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x + 9, y - 4). \end{aligned}$$

(b) The translation $t_{p,q}$ maps the point $(2, 4)$ to $(2 + p, 4 + q)$. For this image to be the point $(4, 2)$, we require $p = 2$ and $q = -2$. Thus the required translation is

$$\begin{aligned} t_{2,-2} : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x + 2, y - 2). \end{aligned}$$

(c) The required rotation is obtained by substituting $\theta = -\frac{3}{4}\pi$ into

$$\begin{aligned} r_\theta : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta). \end{aligned}$$

Since

$$\cos\left(-\tfrac{3}{4}\pi\right) = -\tfrac{1}{2}\sqrt{2} \quad \text{and} \quad \sin\left(-\tfrac{3}{4}\pi\right) = -\tfrac{1}{2}\sqrt{2},$$

the required rotation is

$$\begin{aligned} r_{-3\pi/4} : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto \left(\tfrac{1}{2}\sqrt{2}(-x + y), \tfrac{1}{2}\sqrt{2}(-x - y)\right). \end{aligned}$$

(d) The required reflection is obtained by substituting $\theta = \frac{3}{8}\pi$ into

$$q_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x, y) \longmapsto (x\cos(2\theta) + y\sin(2\theta), x\sin(2\theta) - y\cos(2\theta)).$$

Since $2\theta = \frac{3}{4}\pi$ (i.e. 135°), we have

$$\cos(2\theta) = -\tfrac{1}{2}\sqrt{2} \quad \text{and} \quad \sin(2\theta) = \tfrac{1}{2}\sqrt{2}.$$

The function for the reflection in ℓ is therefore

$$q_{3\pi/8} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x, y) \longmapsto \left(\tfrac{1}{2}\sqrt{2}(-x + y), \tfrac{1}{2}\sqrt{2}(x + y)\right).$$

Solution 2.4

(a) $$\begin{aligned} t_{1,-2} \circ r_{-3\pi/2}(x, y) &= t_{1,-2}(r_{-3\pi/2}(x, y)) \\ &= t_{1,-2}(-y, x) \\ &\qquad (\text{as } r_{-3\pi/2}(x, y) = (-y, x)) \\ &= (-y + 1, x - 2) \end{aligned}$$

(b) $$\begin{aligned} r_{-3\pi/2} \circ t_{1,-2}(x, y) &= r_{-3\pi/2}(t_{1,-2}(x, y)) \\ &= r_{-3\pi/2}(x + 1, y - 2) \\ &= (-(y - 2), (x + 1)) \\ &\qquad (\text{as } r_{-3\pi/2}(x, y) = (-y, x)) \\ &= (-y + 2, x + 1) \end{aligned}$$

(c) $$\begin{aligned} r_{-\pi/2} \circ q_{\pi/4}(x, y) &= r_{-\pi/2}(q_{\pi/4}(x, y)) \\ &= r_{-\pi/2}(y, x) \\ &\qquad (\text{as } q_{\pi/4}(x, y) = (y, x)) \\ &= (x, -y) \\ &\qquad (\text{as } r_{-\pi/2}(x, y) = (y, -x)) \end{aligned}$$

(d) $$\begin{aligned} q_{\pi/4} \circ r_{-\pi/2}(x, y) &= q_{\pi/4}(r_{-\pi/2}(x, y)) \\ &= q_{\pi/4}(y, -x) \\ &= (-x, y) \\ &\qquad (\text{as } q_{\pi/4}(x, y) = (y, x)) \end{aligned}$$

Solution 2.5

(a) Since $t_{-1,3}(x, y) = (x - 1, y + 3)$,

$$\begin{aligned} t_{-1,3}(-2, -1) &= (-2 - 1, -1 + 3) = (-3, 2), \\ t_{-1,3}(-1, 1) &= (-2, 4), \\ t_{-1,3}(2, 0) &= (1, 3). \end{aligned}$$

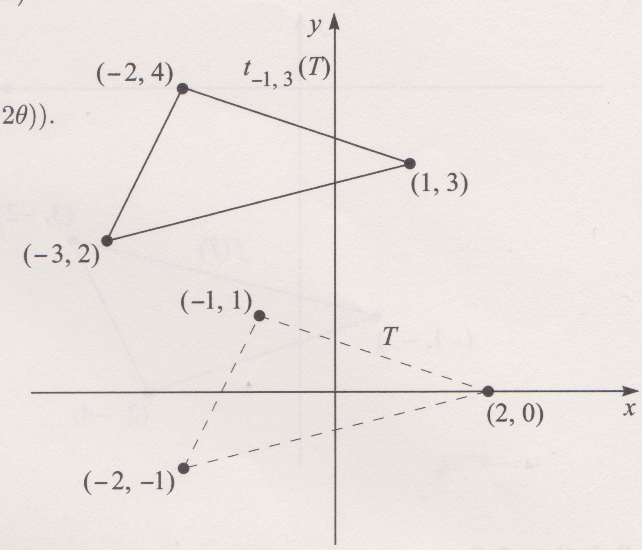
Since $r_\pi(x, y) = (-x, -y)$,

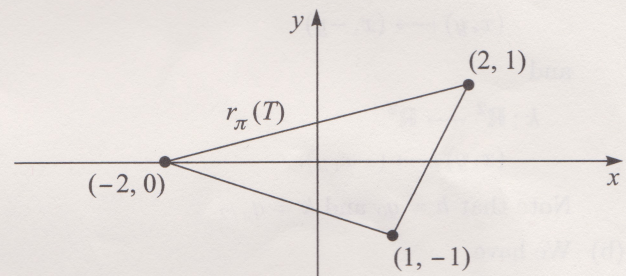
$$\begin{aligned} r_\pi(-2, -1) &= (2, 1), \\ r_\pi(-1, 1) &= (1, -1), \\ r_\pi(2, 0) &= (-2, 0). \end{aligned}$$

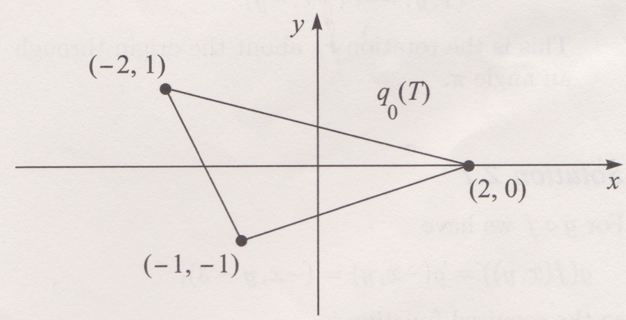
Since $q_0(x, y) = (x, -y)$,

$$\begin{aligned} q_0(-2, -1) &= (-2, 1), \\ q_0(-1, 1) &= (-1, -1), \\ q_0(2, 0) &= (2, 0). \end{aligned}$$

(b)





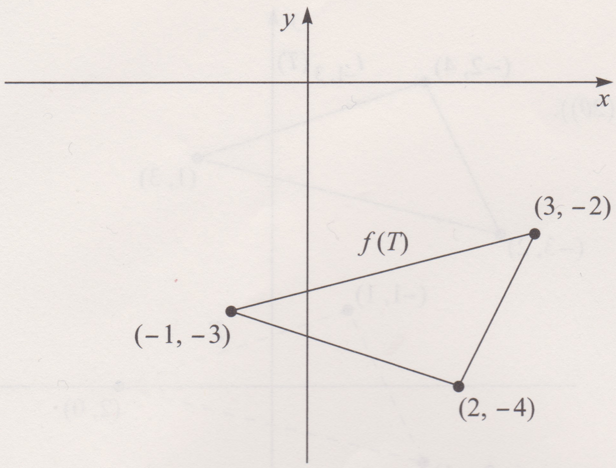


(c) $$\begin{aligned} f(x, y) &= r_\pi \circ t_{-1,3}(x, y) \\ &= r_\pi(t_{-1,3}(x, y)) \\ &= r_\pi(x - 1, y + 3) \\ &= (-(x - 1), -(y + 3)) \\ &= (-x + 1, -y - 3) \end{aligned}$$

Thus

$$\begin{aligned} f(-2, -1) &= (3, -2), \\ f(-1, 1) &= (2, -4), \\ f(2, 0) &= (-1, -3). \end{aligned}$$

(d)



Solution 2.6

(a) First

$$h:\mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x,y) \longmapsto (x,-y)$$

and

$$k:\mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x,y) \longmapsto (-x,y).$$

Note that $h = q_0$ and $k = q_{\pi/2}$.

(b) We have

$$h(k(x,y)) = h(-x,y) = (-x,-y),$$

so the required function is

$$h\circ k:\mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x,y) \longmapsto (-x,-y).$$

This is the rotation r_π about the origin through an angle π.

Solution 2.7

For $g\circ f$ we have

$$g(f(x,y)) = g(-x,y) = (-x,y-3),$$

so the required function is

$$g\circ f:\mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x,y) \longmapsto (-x,y-3).$$

We hope that you can recognise f as reflection in the y-axis; also g is a translation parallel to the y-axis (by three units down). Hence $g\circ f$ is a glide-reflection in the y-axis.

Solution 3.1

We have

$$\begin{aligned}\sin\left(\tfrac{13}{12}\pi\right) &= \sin\left(\tfrac{3}{4}\pi+\tfrac{1}{3}\pi\right)\\ &= \sin\left(\tfrac{3}{4}\pi\right)\cos\left(\tfrac{1}{3}\pi\right)+\cos\left(\tfrac{3}{4}\pi\right)\sin\left(\tfrac{1}{3}\pi\right)\\ &= \left(\tfrac{1}{2}\sqrt{2}\right)\left(\tfrac{1}{2}\right)-\left(\tfrac{1}{2}\sqrt{2}\right)\left(\tfrac{1}{2}\sqrt{3}\right)\\ &= \tfrac{1}{4}\left(\sqrt{2}-\sqrt{6}\right).\end{aligned}$$

Also

$$\begin{aligned}\cos\left(\tfrac{13}{12}\pi\right) &= \cos\left(\tfrac{3}{4}\pi+\tfrac{1}{3}\pi\right)\\ &= \cos\left(\tfrac{3}{4}\pi\right)\cos\left(\tfrac{1}{3}\pi\right)-\sin\left(\tfrac{3}{4}\pi\right)\sin\left(\tfrac{1}{3}\pi\right)\\ &= \left(-\tfrac{1}{2}\sqrt{2}\right)\left(\tfrac{1}{2}\right)-\left(\tfrac{1}{2}\sqrt{2}\right)\left(\tfrac{1}{2}\sqrt{3}\right)\\ &= -\tfrac{1}{4}\left(\sqrt{2}+\sqrt{6}\right),\end{aligned}$$

and

$$\begin{aligned}\tan\left(\tfrac{13}{12}\pi\right) &= \tan\left(\tfrac{3}{4}\pi+\tfrac{1}{3}\pi\right)\\ &= \frac{\tan\left(\frac{3}{4}\pi\right)+\tan\left(\frac{1}{3}\pi\right)}{1-\tan\left(\frac{3}{4}\pi\right)\tan\left(\frac{1}{3}\pi\right)}\\ &= \frac{-1+\sqrt{3}}{1-(-1)\sqrt{3}}\\ &= \frac{-1+\sqrt{3}}{1+\sqrt{3}} = 2-\sqrt{3},\end{aligned}$$

after rationalising the denominator.

Solution 3.2

(a) Putting $\theta = \frac{1}{8}\pi$ into the double-angle formula

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$

gives

$$\tan\left(\tfrac{1}{4}\pi\right) = \frac{2\tan\left(\frac{1}{8}\pi\right)}{1-\tan^2\left(\frac{1}{8}\pi\right)}.$$

As $\tan\left(\frac{1}{4}\pi\right) = 1$, this equation becomes

$$\frac{2\tan\left(\frac{1}{8}\pi\right)}{1-\tan^2\left(\frac{1}{8}\pi\right)} = 1.$$

This can be rearranged to give the following quadratic equation for $\tan\left(\frac{1}{8}\pi\right)$:

$$\tan^2\left(\tfrac{1}{8}\pi\right)+2\tan\left(\tfrac{1}{8}\pi\right)-1=0.$$

The quadratic equation formula then gives

$$\begin{aligned}\tan\left(\tfrac{1}{8}\pi\right) &= \frac{-2\pm\sqrt{2^2-4\times1\times(-1)}}{2}\\ &= \frac{-2\pm\sqrt{8}}{2}\\ &= -1\pm\sqrt{2}\quad(\text{since } \sqrt{8}=2\sqrt{2}).\end{aligned}$$

Since $0<\frac{1}{8}\pi<\frac{1}{2}\pi$, $\tan\left(\frac{1}{8}\pi\right)$ is positive, so we take the positive square root in the formula above and obtain the exact value

$$\tan\left(\tfrac{1}{8}\pi\right) = -1+\sqrt{2}\ (\simeq 0.414).$$

(b) We have to show that

$$\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}} = -1+\sqrt{2}.$$

Since

$$\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}} = \sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}},$$

it is enough to show that

$$\frac{2-\sqrt{2}}{2+\sqrt{2}} = (-1+\sqrt{2})^2.$$

The RHS equals

$$(-1+\sqrt{2})^2 = 3-2\sqrt{2};$$

and the LHS equals

$$\begin{aligned}\frac{2-\sqrt{2}}{2+\sqrt{2}} &= \frac{(2-\sqrt{2})^2}{(2+\sqrt{2})(2-\sqrt{2})}\\ &= \frac{6-4\sqrt{2}}{4-2}\\ &= 3-2\sqrt{2},\end{aligned}$$

which equals the RHS, as required.

Solution 3.3

Since $\cos^2\theta+\sin^2\theta=1$ and $\cos\theta=\frac{1}{3}$, we have

$$\sin^2\theta = 1-\left(\tfrac{1}{3}\right)^2 = \tfrac{8}{9},$$

so

$$\sin\theta = \pm\sqrt{\tfrac{8}{9}} = \pm\tfrac{2}{3}\sqrt{2}.$$

Since θ is in the interval $(-\frac{1}{2}\pi, 0)$, $\sin\theta$ is negative, so we need to take the negative square root; that is,

$$\sin\theta = -\tfrac{2}{3}\sqrt{2}.$$

Then

$$\begin{aligned}\sin(2\theta) &= 2\sin\theta\cos\theta\\ &= 2\times\left(-\tfrac{2}{3}\sqrt{2}\right)\times\tfrac{1}{3}\\ &= -\tfrac{4}{9}\sqrt{2}.\end{aligned}$$

Solution 3.4

Since $\sec(2\theta)=5$, we have

$$\cos(2\theta) = \frac{1}{\sec(2\theta)} = \tfrac{1}{5}.$$

Since θ lies in $(\frac{1}{2}\pi, \pi)$, $\cos\theta$ is negative and $\sin\theta$ is positive, so

$$\cos\theta = -\sqrt{\frac{1+\cos(2\theta)}{2}} = -\sqrt{\frac{1+\frac{1}{5}}{2}} = -\sqrt{\tfrac{3}{5}},$$

and

$$\sin\theta = \sqrt{\frac{1-\cos(2\theta)}{2}} = \sqrt{\frac{1-\frac{1}{5}}{2}} = \sqrt{\tfrac{2}{5}}.$$

Solution 3.5

(a) Putting $\phi=\frac{1}{2}(A+B)$ and $\theta=\frac{1}{2}(A-B)$ in the formulas

$$\begin{aligned}\sin(\phi+\theta) &= \sin\phi\cos\theta+\cos\phi\sin\theta\\ \sin(\phi-\theta) &= \sin\phi\cos\theta-\cos\phi\sin\theta\end{aligned}$$

gives the formulas

$$\begin{aligned}\sin A = {}& \sin(\tfrac{1}{2}(A+B))\cos(\tfrac{1}{2}(A-B))\\ &+\cos(\tfrac{1}{2}(A+B))\sin(\tfrac{1}{2}(A-B)) \quad (1)\\ \sin B = {}& \sin(\tfrac{1}{2}(A+B))\cos(\tfrac{1}{2}(A-B))\\ &-\cos(\tfrac{1}{2}(A+B))\sin(\tfrac{1}{2}(A-B)). \quad (2)\end{aligned}$$

The sum of the LHS of equations (1) and (2) above,

$$\sin A+\sin B,$$

then equals the sum of the RHS, which is

$$2\sin(\tfrac{1}{2}(A+B))\cos(\tfrac{1}{2}(A-B));$$

that is,

$$\sin A+\sin B = 2\sin(\tfrac{1}{2}(A+B))\cos(\tfrac{1}{2}(A-B)),$$

as required.

(b) Instead of taking the sum of the LHS of equation (1) and equation (2), as in the previous part, take their difference,

$$\sin A-\sin B.$$

This difference then equals the difference of the RHS of equation (1) and equation (2), which is

$$2\cos(\tfrac{1}{2}(A+B))\sin(\tfrac{1}{2}(A-B)).$$

Thus

$$\sin A-\sin B = 2\cos(\tfrac{1}{2}(A+B))\sin(\tfrac{1}{2}(A-B)).$$

Solution 3.6

Using the sum formula for $\cos(\phi+\theta)$ with $\phi=2\theta$, we obtain

$$\begin{aligned}\cos(3\theta) &= \cos(2\theta+\theta)\\ &= \cos(2\theta)\cos\theta-\sin(2\theta)\sin\theta\\ &= (2\cos^2\theta-1)\cos\theta-(2\sin\theta\cos\theta)\sin\theta\\ &\quad\text{(using the double-angle formulas)}\\ &= (2\cos^2\theta-1)\cos\theta-2(1-\cos^2\theta)\cos\theta\\ &\quad\text{(using the formula } \sin^2\theta=1-\cos^2\theta\text{)}\\ &= 4\cos^3\theta-3\cos\theta.\end{aligned}$$

Solution 4.1

(a) Here $A=1$, $B=4$ and $C=-1$. The inclination θ of L satisfies

$$\tan(2\theta) = \frac{4}{1-(-1)} = 2,$$

so $\theta=\frac{1}{2}\arctan 2 \simeq 0.554$ radians (approximately $32°$). Since $\tan(2\theta)=2$, the

equations

$$\cos(2\theta) = \frac{1}{\sqrt{1+\tan^2(2\theta)}},$$

$$\sin(2\theta) = \frac{\tan(2\theta)}{\sqrt{1+\tan^2(2\theta)}}$$

give

$$\cos(2\theta) = 1/\sqrt{1+2^2} = \tfrac{1}{\sqrt{5}}$$

and

$$\sin(2\theta) = 2/\sqrt{1+2^2} = \tfrac{2}{\sqrt{5}}.$$

By the half-angle formulas,

$$\cos^2\theta = \tfrac{1}{2}\left(1+\tfrac{1}{\sqrt{5}}\right) = \tfrac{1}{2} + \tfrac{1}{2\sqrt{5}},$$

$$\sin^2\theta = \tfrac{1}{2}\left(1-\tfrac{1}{\sqrt{5}}\right) = \tfrac{1}{2} - \tfrac{1}{2\sqrt{5}},$$

and

$$\sin\theta\cos\theta = \tfrac{1}{2} \times \tfrac{2}{\sqrt{5}} = \tfrac{1}{\sqrt{5}}.$$

Hence, by equations (4.6) in Section 4 of Chapter A3,

$$\begin{aligned} A' &= 1 \times \left(\tfrac{1}{2} + \tfrac{1}{2\sqrt{5}}\right) + 4 \times \tfrac{1}{\sqrt{5}} - 1 \times \left(\tfrac{1}{2} - \tfrac{1}{2\sqrt{5}}\right) \\ &= 5 \times \tfrac{1}{\sqrt{5}} = \sqrt{5}, \\ C' &= 1 \times \left(\tfrac{1}{2} - \tfrac{1}{2\sqrt{5}}\right) - 4 \times \tfrac{1}{\sqrt{5}} - 1 \times \left(\tfrac{1}{2} + \tfrac{1}{2\sqrt{5}}\right) \\ &= -5 \times \tfrac{1}{\sqrt{5}} = -\sqrt{5}. \end{aligned}$$

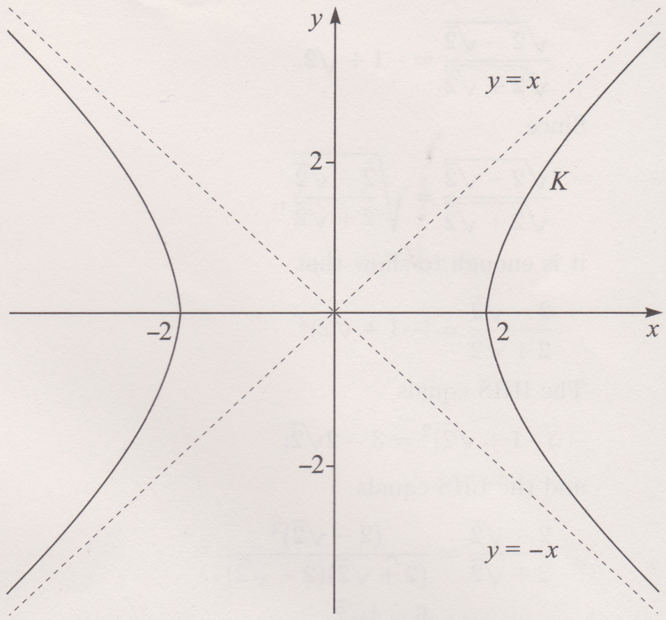
Since $F' = F = -4\sqrt{5}$, the equation of K is

$$\sqrt{5}x^2 - \sqrt{5}y^2 - 4\sqrt{5} = 0; \quad \text{that is,} \quad \frac{x^2}{4} - \frac{y^2}{4} = 1.$$

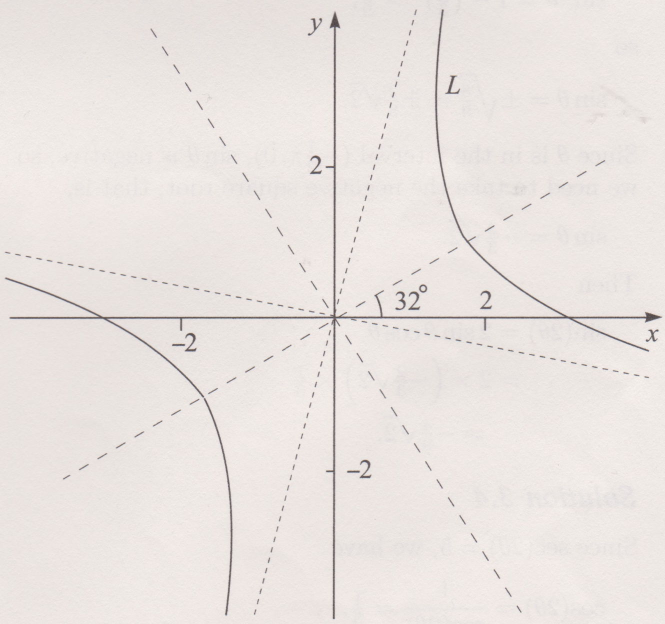
Hence K is a hyperbola in standard position with $a = 2$ and $b = 2$, with vertices at $(2, 0)$ and $(-2, 0)$ and asymptotes

$$y = \pm\frac{2}{2}x = \pm x,$$

as follows.



L is obtained by rotating K anticlockwise by approximately 32°. (Note that 32° is a little more than one-third of a right angle.) The sketch below includes the axes of symmetry of L and its asymptotes (although the latter are not essential).



(b) Here $A = 8$, $B = -4\sqrt{3}$ and $C = 4$. The inclination θ of L satisfies

$$\tan(2\theta) = \frac{-4\sqrt{3}}{8-4} = -\sqrt{3},$$

so $\theta = \frac{1}{2}\arctan(-\sqrt{3}) = -\frac{1}{6}\pi$ (= −30°). Thus

$$\cos^2\theta = \cos^2\left(-\tfrac{1}{6}\pi\right) = \left(\tfrac{1}{2}\sqrt{3}\right)^2 = \tfrac{3}{4}$$

$$\sin^2\theta = \sin^2\left(-\tfrac{1}{6}\pi\right) = \left(-\tfrac{1}{2}\right)^2 = \tfrac{1}{4},$$

and

$$\sin\theta\cos\theta = \tfrac{1}{2}\sqrt{3} \times \left(-\tfrac{1}{2}\right) = -\tfrac{1}{4}\sqrt{3}.$$

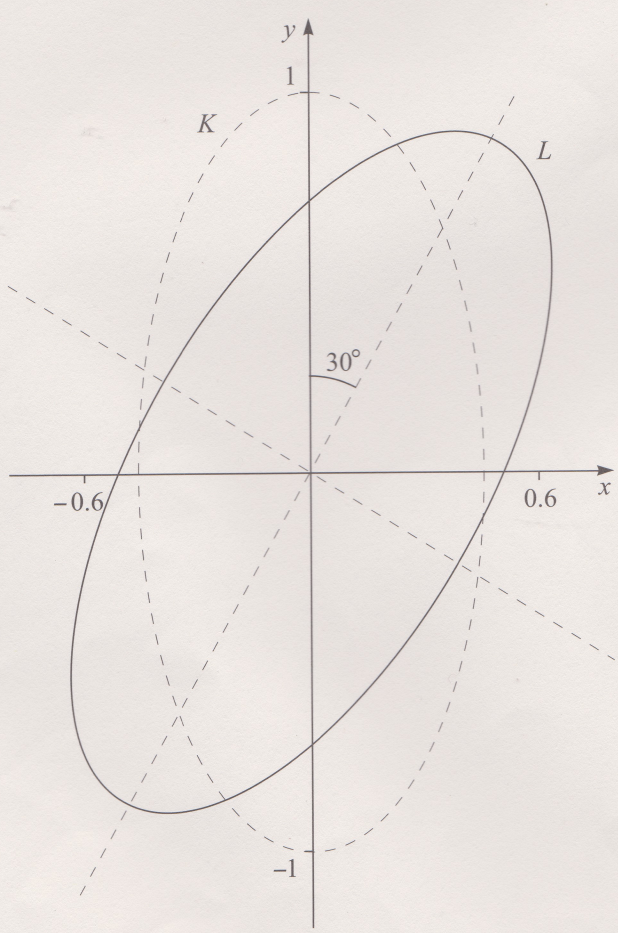
Hence, by equations (4.6) in Section 4 of Chapter A3,

$$\begin{aligned} A' &= 8 \times \tfrac{3}{4} + (-4\sqrt{3}) \times \left(-\tfrac{1}{4}\sqrt{3}\right) + 4 \times \tfrac{1}{4} \\ &= 6 + 3 + 1 = 10, \\ C' &= 8 \times \tfrac{1}{4} - (-4\sqrt{3}) \times \left(-\tfrac{1}{4}\sqrt{3}\right) + 4 \times \tfrac{3}{4} \\ &= 2 - 3 + 3 = 2. \end{aligned}$$

Since $F' = F = -2$, the equation of K is

$$10x^2 + 2y^2 - 2 = 0; \quad \text{that is,} \quad \frac{x^2}{1/5} + y^2 = 1.$$

Hence K is an ellipse in reflected standard position with $a = \frac{1}{\sqrt{5}} \simeq 0.45$ and $b = 1$, and L is obtained by rotating K clockwise by $30°$.



(c) Here $A = 3$, $B = 2$ and $C = 3$. The inclination θ of L is $\frac{1}{4}\pi$, since $A = C = 3$, so

$$\cos^2\theta = \sin^2\theta = \sin\theta\cos\theta = \tfrac{1}{2}.$$

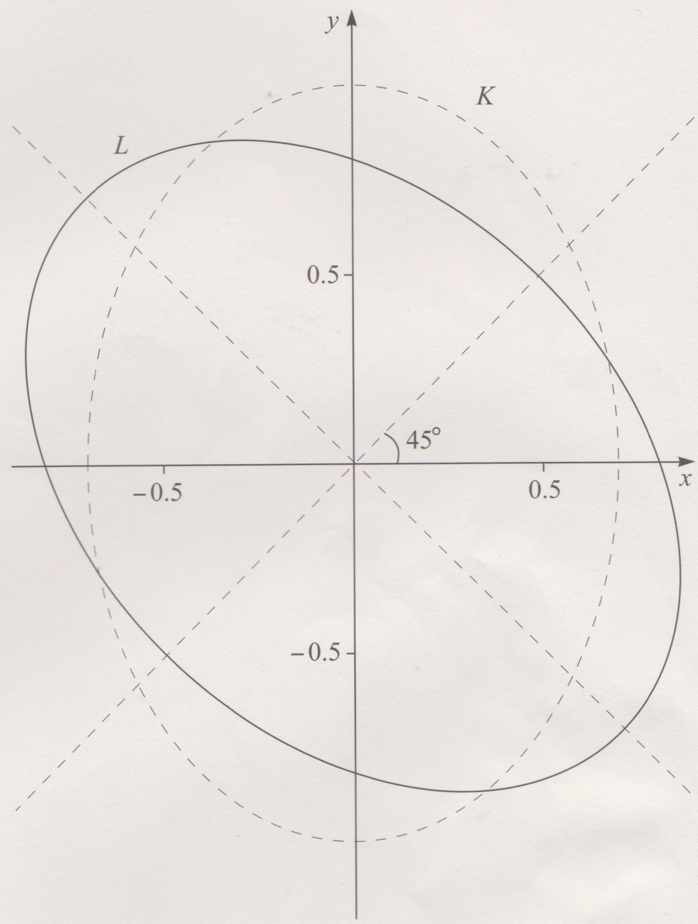
Hence, by equations (4.6) in Section 4 of Chapter A3,

$$\begin{aligned} A' &= 3 \times \tfrac{1}{2} + 2 \times \tfrac{1}{2} + 3 \times \tfrac{1}{2} = 4, \\ C' &= 3 \times \tfrac{1}{2} - 2 \times \tfrac{1}{2} + 3 \times \tfrac{1}{2} = 2. \end{aligned}$$

Since $F' = F = 2$, the equation of K is

$$4x^2 + 2y^2 - 2 = 0, \quad \text{that is,} \quad \frac{x^2}{1/2} + y^2 = 1.$$

Hence K is an ellipse in reflected standard position with $a = \frac{1}{\sqrt{2}} \simeq 0.71$ and $b = 1$, and L is obtained by rotating K anticlockwise by $\frac{1}{4}\pi$ radians.



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